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On the global version of Euler–Lagrange equations

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Abstract

The introduction of a covariant derivative on the velocity phase space is needed for a global expression of Euler–Lagrange equations. The aim of this paper is to show how its torsion tensor turns out to be involved in such a version.

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An increasing attention has been recently paid to coordinate-free formulations of motion equations in classical mechanics (see for instance [1, 2] and references therein). In this work we write down intrinsic Euler–Lagrange equations and show the appearance of a torsion term. Furthermore, we shall see that this term should also be present in the horizontal Lagrange–Poincaré equations considered in [1, 2], if the torsion of the chosen derivative does not vanish.

It is worth noting that covariant derivatives with non-vanishing torsion naturally arise in several branches of physics; namely dynamics with nonholonomic constraints [3, 4], E Cartan's theory of gravity (see for instance [5]) and modern string theories (see for example [6]), among others.

Let us consider a physical system with configuration manifold Q and Lagrangian $L(q, \dot{q}) : TQ \to \mathbb{R}$ (for this geometrical setting, see for instance [7]).

If a coordinate-free characterization of the Euler–Lagrange equations associated with the system is required a covariant derivative D must be introduced to TQ, for $\frac{\partial L}{\partial q}$ is involved (see for instance [8]). Once such D is chosen, $\frac{\mathrm{D}L}{\mathrm{D}q}$ is defined in the standard way

$$\frac{\mathrm{D}L}{\mathrm{D}q}(q_0, \dot{q}_0) = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} L \circ \gamma(\lambda) \tag{1}$$

with $\gamma(\lambda) = (q(\lambda), \dot{q}_{0\|}(\lambda))$, $q(0) = q_0, \dot{q}(0) = \dot{q}_0$ and $\dot{q}_{0\|}(\lambda)$ the parallel transport of $\dot{q}_0$ along $q(\lambda)$.

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Moreover, an associated covariant derivative on T^*Q, that we will also denote by D, is naturally defined through the Leibnitz rule: for any curves $\alpha(t)$ and $v(t)$ in T^*Q and TQ, respectively

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\alpha(t), v(t)\rangle = \left\langle\frac{\mathrm{D}\alpha(t)}{\mathrm{D}t}, v(t)\right\rangle + \left\langle\alpha(t), \frac{Dv(t)}{Dt}\right\rangle \tag{2}$$

where $\langle\, ,\, \rangle$ denotes the pairing between T^*Q and TQ.

It is worth noting that $\frac{\partial}{\partial\dot{q}}$ has a coordinate-free sense: it is the derivative along the fibre.

Proposition 1. *Let D be an arbitrary covariant derivative on TQ. Then the coordinate-free expression of the Euler–Lagrange equations is*

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial L}{\partial\dot{q}}\right) - \frac{\mathrm{D}L}{\mathrm{D}q} = \frac{\partial L}{\partial\dot{q}}T(\dot{q}(t),\,) \tag{3}$$

where $T(\, ,\,)$ is the torsion tensor of D.

Proof. The curve $q(t)$ is a solution of the Euler–Lagrange equations if and only if it is a critical point for the action

$$S = \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\,\mathrm{d}t \tag{4}$$

for variations of the curves such that q_0 and q_1 remain fixed. That is, for each $q(t,\lambda)$: $[t_0, t_1] \times (-\varepsilon, \varepsilon) \to Q$ such that $q(t, 0) = q(t)$, $q(t_0, \lambda) = q(t_0)$ and $q(t_1, \lambda) = q(t_1)$,

$$\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} \int_{t_0}^{t_1} L(q(t,\lambda), \dot{q}(t,\lambda))\,\mathrm{d}t = \int_{t_0}^{t_1} \delta L(q(t), \dot{q}(t))\,\mathrm{d}t = 0 \tag{5}$$

where $\delta L(q(t), \dot{q}(t)) = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} L(q(t,\lambda), \dot{q}(t,\lambda))$.

But

$$\begin{aligned}\delta L(q(t), \dot{q}(t)) &= \lim_{\lambda\to 0}\frac{L(q(t,\lambda), \dot{q}(t,\lambda)) - L(q(t,0), \dot{q}(t,0))}{\lambda}\\ &= \lim_{\lambda\to 0}\frac{L(q(t,\lambda), \dot{q}(t,\lambda)) - L(q(t,\lambda), \dot{q}_{\|}(t,\lambda))}{\lambda}\\ &\quad + \lim_{\lambda\to 0}\frac{L(q(t,\lambda), \dot{q}_{\|}(t,\lambda)) - L(q(t,0), \dot{q}(t,0))}{\lambda}\end{aligned} \tag{6}$$

where $\dot{q}_{\|}(t,\lambda)$ is the parallel translate of the vector $\dot{q}(t, 0)$ along the curve $q(t,\lambda)|_{\text{fixed } t}$ (see figure 1).

Then

$$\delta L(q(t), \dot{q}(t)) = \frac{\partial L}{\partial\dot{q}}\mathrm{D}_{\delta q(t)}\dot{q}(t) + \frac{\mathrm{D}L}{\mathrm{D}q}\delta q(t) \tag{7}$$

where we have denoted

$$\delta q(t) = \left.\frac{\partial q(t,\lambda)}{\partial\lambda}\right|_{\lambda=0} \qquad \text{and} \qquad \dot{q}(t) = \left.\frac{\partial q(t,\lambda)}{\partial t}\right|_{\lambda=0}. \tag{8}$$

By definition of the torsion tensor $T(\, ,\,)$, we have

$$T(\dot{q}(t), \delta q(t)) = \mathrm{D}_{\dot{q}(t)}\delta q(t) - \mathrm{D}_{\delta q(t)}\dot{q}(t) - [\dot{q}(t), \delta q(t)]. \tag{9}$$

Thus, by using (2) and taking into account that $[\dot{q}(t), \delta q(t)]$ vanishes, we have

$$\begin{aligned}\delta L(q(t), \dot{q}(t)) &= \frac{\partial L}{\partial\dot{q}}\mathrm{D}_{\dot{q}(t)}\delta q(t) + \frac{\mathrm{D}L}{\mathrm{D}q}\delta q(t) - \frac{\partial L}{\partial\dot{q}}T(\dot{q}(t), \delta q(t))\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}}\delta q(t)\right) - \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial L}{\partial\dot{q}}\right)\delta q(t) + \frac{\mathrm{D}L}{\mathrm{D}q}\delta q(t) - \frac{\partial L}{\partial\dot{q}}T(\dot{q}(t), \delta q(t)).\end{aligned} \tag{10}$$

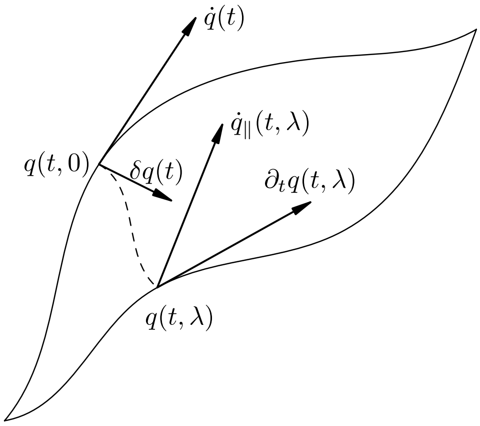

Figure 1. The vector $\dot{q}_{\|}(t,\lambda)$ is the parallel transport of the vector $\dot{q}(t,0)$ along the dashed curve.

Now integrating along the curve $q(t)$ we finally get

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q(t) - \frac{\mathrm{D}L}{\mathrm{D}q}\delta q(t) = \frac{\partial L}{\partial \dot{q}} T(\dot{q}(t), \delta q(t)). \tag{11}$$

□

Remark 1. Of course, regardless of the covariant derivative D we introduced, in *any* coordinate patch the Euler–Lagrange equations always read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0. \tag{12}$$

Remark 2. As well known (e.g., [8]), a torsion-free D can always be chosen. It is obvious that for such a connection, global Euler–Lagrange equations read

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\mathrm{D}L}{\mathrm{D}q} = 0. \tag{13}$$

So in this case, the global expression can be obtained merely by replacing the usual derivatives by D in (12).

Remark 3. A similar result holds for the horizontal Lagrange–Poincaré equations considered in [1, 2]. One of the goals of these references is to analyse the intrinsic meaning of motion equations for constrained systems with symmetries. Let us recall that, in such a system, the Lagrangian L and the constraints remain invariant under the lifting to TQ of a suitable action of a Lie group G on Q. A connection A, related to the constraints, is introduced on the principal bundle $Q \stackrel{\pi}{\to} Q/G$. If $\tilde{\mathfrak{g}}$ is the adjoint bundle to the principal bundle Q, the connection A yields an isomorphism α between TQ/G and the Whitney sum $T(Q/G) \oplus \tilde{\mathfrak{g}}$ in the following way

$$\alpha_A[q, \dot{q}]_G = (x, \dot{x}, \tilde{v}) = \pi_*(q, \dot{q}) \oplus [q, A(q, \dot{q})]_G. \tag{14}$$

Now, one can define the reduced Lagrangian $\ell : T(Q/G) \oplus \tilde{\mathfrak{g}} \to \mathbb{R}$ as

$$\ell(x, \dot{x}, \tilde{v}) = L(q, \dot{q}). \tag{15}$$

A variation δq of a curve in Q is said to be *horizontal* if $A(\delta q) = 0$. In this case, the corresponding variation $\alpha(\delta q(t))$ of the curve $\alpha(q(t))$ in $T(Q/G) \oplus \tilde{\mathfrak{g}}$ is [1]

$$\alpha(\delta q(t)) = \delta x \oplus \tilde{B}(\delta x, \dot{x}) \tag{16}$$

where $\tilde{B}$ is the $\tilde{\mathfrak{g}}$-valued two-form on Q/G defined by

$$\tilde{B}([q]_G)(X, Y) = [q, B(X^h(q), Y^h(q))]_G \tag{17}$$

with X^h, Y^h the horizontal lifts to Q of X and Y, and B the curvature of the connection A.

The horizontal Lagrange–Poincaré equations for L are defined as the Euler–Lagrange ones for ℓ restricted to horizontal variations δq. A coordinate-free version of them can be written down by introducing an arbitrary covariant derivative D on $T(Q/G)$ and using the covariant derivative $\tilde{\mathrm{D}}$ induced by A on $\tilde{\mathfrak{g}}$.

Under the implicit assumption that the torsion of D vanishes, it is shown in [1, 2] that, for horizontal variations δq,

$$\delta \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\, \mathrm{d}t = 0 \tag{18}$$

if and only if the following horizontal Lagrange–Poincaré equations hold

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial \ell}{\partial \dot{x}}\right)(x, \dot{x}, \tilde{v}) - \frac{\mathrm{D}\ell}{\mathrm{D}x}(x, \dot{x}, \tilde{v}) = -\left\langle \frac{\partial \ell}{\partial \tilde{v}}, \tilde{B}(x)(\dot{x}, .)\right\rangle. \tag{19}$$

It is easy to see that, for an arbitrary covariant derivative D on $T(Q/G)$, its torsion tensor T must be taken into account in the previous equations. Arguing as above one gets

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial \ell}{\partial \dot{x}}\right)(x, \dot{x}, \tilde{v}) - \frac{\mathrm{D}\ell}{\mathrm{D}x}(x, \dot{x}, \tilde{v}) = -\left\langle \frac{\partial \ell}{\partial \tilde{v}}, \tilde{B}(x)(\dot{x}, .)\right\rangle + T(\dot{q}, .). \tag{20}$$

Assuming as in [1, 2] the torsion-free requirement for D, the last term clearly vanishes and we recover the horizontal Lagrange–Poincaré equations found in those references.

Again, in *any* coordinate patch, the expression of horizontal Lagrange–Poincaré equations is independent of the choice of the covariant derivative D.

Remark 4. When considered as a map of the second-order tangent bundle $T(TQ)$ to T^*Q, the Euler–Lagrange operator turns out to be intrinsic without any choice of connection (e.g., [9]). The need for connections appears if one prefers to stay in the framework of tangent bundles, as it is usually done, and not to deal with second-order ones.

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